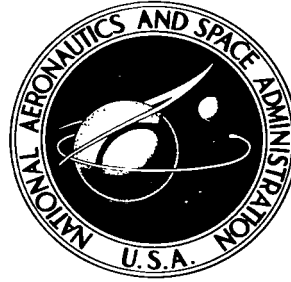


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TRIAxIAL BALANCING TECHNIQUES

(A STUDY OF SPACECRAFT BALANCE
WITH RESPECT TO MULTIPLE AXES)

by William E. Lang

*Goddard Space Flight Center
Greenbelt, Maryland*

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1964



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SUMMARY

Spacecraft unbalance tolerance is likely to be expressed in terms of displacement of the center of gravity from its nominal position, and angular deviation between the principal and reference axes. This study discusses the relations that define the unbalance of a spacecraft, with respect to three mutually perpendicular reference axes, in terms of measurable mass parameters. It was motivated by the need to develop practical methods for balancing the San Marco spacecraft (the San Marco project is a joint effort of Italy and the United States). The theory proved directly and effectively applicable; but complications due to the inaccuracies of measured input data necessitated the development of modified methods for computing corrections.

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INTRODUCTION

The meaning and significance of static and dynamic balance of spin-stabilized spacecraft is well defined, and systematic means of measuring and controlling it have been developed.* This kind of balancing, which may be called "single axis balancing," requires that a reference axis (the spin axis) be a principal axis of inertia of the spacecraft and that the spacecraft center of gravity be on this axis.

Some future spacecraft, because of active attitude control systems or other operational considerations, will require extension of this concept to include concurrent static and dynamic balancing with respect to three mutually perpendicular reference axes. This may be called "triaxial balancing."

Triaxial balancing requires that the center of gravity coincides with the intersection of three mutually perpendicular reference axes, and that these reference axes coincide with the principal axes of inertia of the spacecraft. Single axis balancing is necessary but not sufficient for triaxial balance. It is sufficient that static and dynamic balance exists about two of the reference axes—balance about the third axis is then automatic.

Under this concept of triaxial balancing measured values of the magnitude and phase of static unbalance and of dynamic unbalance will be used to compute the correction needed to balance a spacecraft triaxially. Correction is assumed to be made by adding weight at the surface of a sphere of unit radius with its center at $x = y = z = 0$. The techniques described in this paper are limited to spacecraft which are essentially rigid, solid, and of constant mass.

DEFINITIONS AND COORDINATE SYSTEM CONVENTIONS

Assume a spacecraft of any size and shape with three reference axes, XX, YY, and ZZ, and any point in space, x, y, z , in a rectangular cartesian coordinate system. An imaginary sphere of

*Schaller, N. C., and Lewallen, J. M., "Methods of Expressing Mass Unbalance," NASA Technical Note D-1446, May 1963.

radius R with its center at $x = 0$, $y = 0$, $z = 0$, would have its surface defined by $x^2 + y^2 + z^2 = R^2$. The unbalance of the spacecraft can be defined in terms of the masses and locations of correction weights theoretically concentrated at points on the sphere surface in order to reduce the product of inertia about all three axes to zero.

This study indicates that either five or three such weights are necessary. More than five could be resolved into five or three; less than three would suffice only in special cases. No rigorous proof is offered for this hypothesis, but it can be supported by considering imaginary situations, and the derivations to be given here that follow from it are logical and consistent.

One weight would correct static unbalance, i.e., center of gravity displacement. It would be located at the sphere surface on a radial straight line from the actual center of gravity passing through the desired center of gravity, $x = y = z = 0$. The other weights would be either one or two diametrically opposed equal pairs. The location of any point can be defined in terms of phase orientation with respect to two of the coordinate axes, plus radial distance from $x = y = z = 0$. Static or dynamic unbalance about XX , YY , and ZZ can also be defined by a vector having magnitude in appropriate units and a phase orientation.

To use both rectangular and polar coordinates, we must relate the two systems by appropriate conventions, which are to some extent arbitrary. Suppose the sphere to be enclosed by a cube and

the cube, unfolded, to appear as shown in Figure 1. The large dots designate the positive directions of axes passing through the centers of the six faces, and the curved arrows indicate phase angle convention. The tail of each arrow is at 0 degrees. The system follows trigonometric and right-hand vector rule convention; i.e., when one is looking along an axis toward the origin from a positive end angles increase counterclockwise. The symbols α , β , and γ will be defined later.

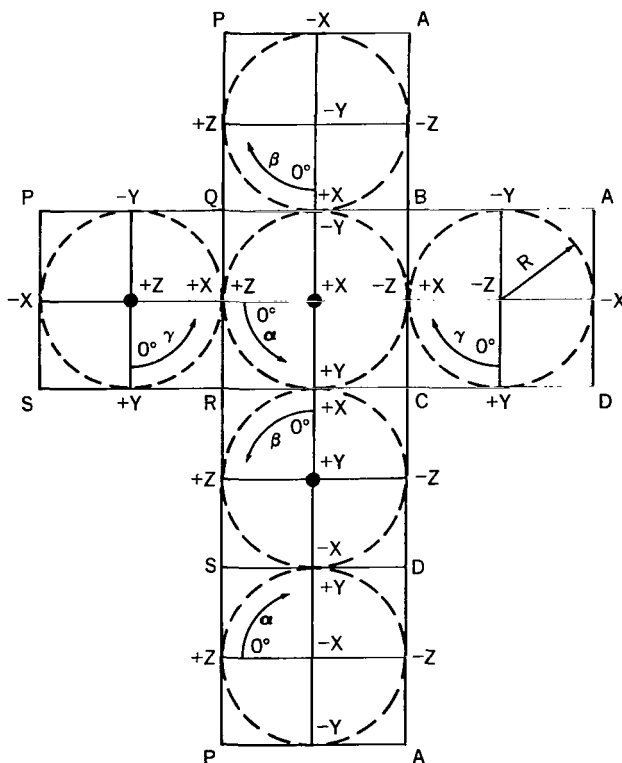


Figure 1—Projection diagram of the coordinate system.

The significance of unbalance phase must be discussed. Static unbalance phase means the angle at which correction by weight addition is indicated. Dynamic unbalance phase also means an angle at which correction by weight addition is indicated. However, this correction has to be made with two weights 180 degrees apart. One system to eliminate ambiguity, is to express dynamic unbalance as a torque vector in a right- or left-hand convention. Instead of this, here the phase of

dynamic unbalance will mean the angle at which the correction weight having a positive coordinate parallel to the reference axis should be located. For example, if dynamic unbalance of $D_x/\underline{\alpha}$ needs weight W at $+x, +y, +z$ plus weight W at $-x, -y, -z$, then α is the phase about the XX axis of the weight at $+x, +y, +z$. (The weight at $-x, -y, -z$ will have a phase angle of $\alpha + 180$ degrees, but the dynamic unbalance phase, by convention, is α). At this point it is necessary to define the following symbols:

$D_x/\underline{\alpha}, D_y/\underline{\beta}, D_z/\underline{\gamma}$ = magnitude and phase of dynamic unbalances about axes XX, YY, ZZ after correction of all static unbalances about $x = y = z = 0$,

I_x, I_y, I_z = spacecraft inertias about XX, YY , and ZZ axes,

I_{xy}, I_{yz}, I_{zx} = products of inertia with respect to indicated subscripts,

M = spacecraft weight,

N and x_N, y_N, z_N = mass and coordinates of static correction weight,

$S_x/\underline{a}, S_y/\underline{b}, S_z/\underline{c}$ = magnitude and phase of static unbalance about axes XX, YY , and ZZ , respectively,

W, x, y, z and $W, -x, -y, -z$ = mass and coordinates of dynamic unbalance correction weights,

x_M, y_M, z_M = coordinates of the spacecraft center of gravity, before correction,

$\delta = \sqrt{x_M^2 + y_M^2 + z_M^2}$, the center of gravity displacement, before correction,

$\theta_x, \theta_y, \theta_z$ = allowable angular deviations between the principal axes and reference axes.

For the mathematical model defined by the foregoing, many relationships are more or less apparent, and the more trivial of these will be stated without explanation or proof.

DEVELOPMENT OF SIGNIFICANT RELATIONSHIPS

Static Unbalance

The indicated correction N at x_N, y_N, z_N for center of gravity displacement is defined by $NR = M\delta, Nx_N = -Mx_M, Ny_N = -My_M$, and $Nz_N = -Mz_M$. From basic concepts or by definition:

$$S_x = N\sqrt{y_N^2 + z_N^2}, \quad (1)$$

$$S_y = N\sqrt{z_N^2 + x_N^2}, \quad (2)$$

$$S_z = N\sqrt{x_N^2 + y_N^2}, \quad (3)$$

$$\tan a = \frac{y_N}{z_N}, \quad (4)$$

$$\tan b = \frac{z_N}{x_N}, \quad (5)$$

$$\tan c = \frac{x_N}{y_N}, \quad (6)$$

$$x_N^2 + y_N^2 + z_N^2 = R^2, \quad (7)$$

$$RN = 0.707\sqrt{S_X^2 + S_Y^2 + S_Z^2}. \quad (8)$$

RN might be called the triaxial static unbalance. Of course R is an assigned value, which could be unity. From Equations 4, 5, and 6, $\tan a \tan b \tan c = 1$; therefore any two phase angles define the third. Equation 4 implies that $\sin a = y_N/\sqrt{y_N^2 + z_N^2}$; therefore $\sqrt{y_N^2 + z_N^2} = y_N/\sin a$. Similarly $\sqrt{z_N^2 + x_N^2} = z_N/\sin b$ and $\sqrt{x_N^2 + y_N^2} = x_N/\sin c$. Therefore Equations 1, 2, and 3 become:

$$N = \frac{S_X \sin a}{y_N} = \frac{S_Y \sin b}{z_N} = \frac{S_Z \sin c}{x_N}, \quad (9)$$

and N must be positive. Equations 4, 5, and 7, or 5, 6, and 7, or 4, 6, and 7 can be solved simultaneously for x_N , y_N , and z_N :

$$\pm x_N = y_N \tan c = \frac{z_N}{\tan b} = \frac{R \cos b \tan c}{\sqrt{\cos^2 b + \tan^2 c}}, \quad (10)$$

$$\pm y_N = z_N \tan a = \frac{x_N}{\tan c} = \frac{R \cos c \tan a}{\sqrt{\cos^2 c + \tan^2 a}}, \quad (11)$$

$$\pm z_N = x_N \tan b = \frac{y_N}{\tan a} = \frac{R \cos a \tan b}{\sqrt{\cos^2 a + \tan^2 b}}. \quad (12)$$

An alternative set of equations, easier to use, is:

$$N = \sqrt{\frac{1}{2}(S_X^2 + S_Y^2 + S_Z^2)}, \quad (13)$$

$$x_N = \sqrt{1 - \frac{S_X^2}{N^2}}, \quad (14)$$

$$y_N = \sqrt{1 - \frac{S_Y^2}{N^2}}, \quad (15)$$

$$z_N = \sqrt{1 - \frac{S_Z^2}{N^2}}. \quad (16)$$

Note that, in Equations 10-12 and 14-16, x_N , y_N , and z_N are positive or negative as necessary to make N positive in Equation 9. This means that x_N has to have the same sign as $\sin c$, y_N the same sign as $\sin a$, and z_N the same sign as $\sin b$.

It may be preferable to use three weights, one on each reference axis, rather than N at x_N , y_N , z_N . (This would not introduce dynamic unbalance as a result of static unbalance correction.) The corrections at unit spherical radius would be

$$S_Y \cos b = S_Z \sin c \text{ at } x = +1, y = z = 0, \quad (17)$$

$$S_X \sin a = S_Z \cos c \text{ at } y = +1, x = z = 0, \quad (18)$$

$$S_X \cos a = S_Y \sin b \text{ at } z = +1, y = x = 0; \quad (19)$$

negative values imply equal positive values at x , y , or $z = -1$. The utility of Equations 9-19 depends on which of the parameters S_X , S_Y , S_Z , a , b , and c are known. In general, if enough parameters to physically define the situation are known, the rest can be calculated.

R , N , x_N , y_N , and z_N define a mass moment with respect to $x = y = z = 0$ (i.e., RN) and the direction cosines of a straight line radiant from $x = y = z = 0$. R is an arbitrary choice, and N varies inversely with R , but the radiant orientation is fixed and independent of R ; i.e., N must lie somewhere along this fixed radiant. Of course two (or more) weights could be used instead of N . Two such weights N_1 and N_2 could be at different radii along the radiant, such that $N_1 R_1 + N_2 R_2 = NR$. They could also be placed at the same radius R along two secondary radiants, each inclined at angle θ to the primary radiant, with the three radiants coplanar, such that $N_1 = N_2$ and $2N_1 \cos \phi = N$. Addition of vectors radiating from a point in three-dimensional space is inherently more complex than the addition of coplanar vectors, because each vector has three components rather than two, but the general procedures are similar.

Dynamic Unbalance

Once the spacecraft center of gravity is at $x = y = z = 0$, (after addition of N) there is no static unbalance. Any remaining unbalance is dynamic, can be expressed as a mass moment couple, and must be corrected by a mass moment couple, or at least the correction must not reintroduce static unbalance. For the "spherical surface correction," this means W at x , y , z plus W at $-x$, $-y$, $-z$. W , x , y , and z are to be determined.

Note that D_X/α , D_Y/β , and D_Z/γ are dynamic unbalances *after* the correction of all static unbalance. This is because the correction N will contribute dynamic unbalance, unless it lies exactly on axis XX , YY , or ZZ . The contributions of dynamic unbalance about these axes due to N are:

$$N_{XX} = Nx \sqrt{y^2 + z^2} / \tan^{-1} \frac{y}{z} + 180^\circ, \quad (20)$$

$$N_{YY} = N_y \sqrt{z^2 + x^2} \angle \tan^{-1} \frac{z}{x} + 180^\circ , \quad (21)$$

$$N_{ZZ} = N_z \sqrt{x^2 + y^2} \angle \tan^{-1} \frac{x}{y} + 180^\circ . \quad (22)$$

Note that the signs of x , y , and z need to be considered in defining phase angles; for example, although $+2/-2 = -2/+2 = -1$, $\tan^{-1}(+2/-2) = 135$ degrees and $\tan^{-1}(-2/+2) = 315$ degrees. The phase angles defined will be those of corrections having positive "parallel to reference axis" coordinates; for example, phase of unbalance about XX is that of a correction weight having a positive x coordinate. W , x , y , and z can be determined in terms of the parameters D_x , D_y , D_z , α , β , and γ and an assigned value for R . Enough parameters to physically define a situation will suffice for a complete solution; for instance, the dynamic unbalance magnitude about one axis, plus the phase of unbalance about any two axes, is sufficient. Therefore any additional data would be redundant. This has significance for computer programming because input must be sufficient but must not be redundant, unless the program provides for using redundant data to compute alternative results. The added data would also be inconsistent insofar as it would not be 100 percent accurate.

From basic concepts or by definition:

$$\tan \alpha = \frac{y}{z} , \quad (23)$$

$$\tan \beta = \frac{z}{x} , \quad (24)$$

$$\tan \gamma = \frac{x}{y} , \quad (25)$$

$$x^2 + y^2 + z^2 = R^2 , \quad (26)$$

$$D_x = 2Wx \sqrt{y^2 + z^2} , \quad (27)$$

$$D_y = 2Wy \sqrt{z^2 + x^2} , \quad (28)$$

$$D_z = 2Wz \sqrt{x^2 + y^2} . \quad (29)$$

Equations 23-25 and 27-29 can be combined into:

$$W = \frac{D_x \sin \alpha}{2xy} = \frac{D_y \sin \beta}{2yz} = \frac{D_z \sin \gamma}{2zx} . \quad (30)$$

Equations 23-26 can be solved for x , y , and z . The results are:

$$\pm x = y \tan \gamma = \frac{z}{\tan \beta} = \frac{R \cos \beta \tan \gamma}{\sqrt{\cos^2 \beta + \tan^2 \gamma}} , \quad (31)$$

$$\pm y = z \tan \alpha = \frac{x}{\tan \gamma} = \frac{R \cos \gamma \tan \alpha}{\sqrt{\cos^2 \gamma + \tan^2 \alpha}} , \quad (32)$$

$$\pm z = x \tan \beta = \frac{y}{\tan \alpha} = \frac{R \cos \alpha \tan \beta}{\sqrt{\cos^2 \alpha + \tan^2 \beta}} . \quad (33)$$

This solution, although correct, proved unsuitable for numerical calculation, because the trigonometric functions involved can have values which are affected drastically by minor deviations of phase angles. A more useful solution, which involves the use of Equations 23-29, is:

$$|w| = \frac{D_x}{2x \sqrt{y^2 + z^2}} = \frac{D_y}{2y \sqrt{z^2 + x^2}} = \frac{D_z}{2z \sqrt{x^2 + y^2}} , \quad (34)$$

$$|x| = \sqrt{1 + \left(\frac{-D_x^2 + D_y^2 + D_z^2}{+D_x^2 - D_y^2 + D_z^2} \right) + \left(\frac{-D_x^2 + D_y^2 + D_z^2}{+D_x^2 + D_y^2 - D_z^2} \right)} , \quad (35)$$

$$|y| = \sqrt{1 + \left(\frac{+D_x^2 - D_y^2 + D_z^2}{-D_x^2 + D_y^2 + D_z^2} \right) + \left(\frac{+D_x^2 - D_y^2 + D_z^2}{+D_x^2 + D_y^2 - D_z^2} \right)} , \quad (36)$$

$$|z| = \sqrt{1 + \left(\frac{+D_x^2 + D_y^2 - D_z^2}{-D_x^2 + D_y^2 + D_z^2} \right) + \left(\frac{+D_x^2 + D_y^2 - D_z^2}{+D_x^2 - D_y^2 + D_z^2} \right)} = 1 - x^2 - y^2 . \quad (37)$$

The values for x , y , and z can be real only if the sum of any two of the quantities D_x^2 , D_y^2 , and D_z^2 exceeds the third quantity. Any real mass configuration will satisfy this condition in fact, but erroneous measurements may not. Methods for "normalizing" measured data to overcome this problem will be discussed later. The method of solving Equations 26-29 for Equations 35-37 is given in Appendix A.

To calculate w , x , y , and z from Equations 30-33 it is necessary to know the dynamic unbalance about one axis, plus the phase of unbalance about any two axes. As R may have any convenient value, it is logical to let R equal one unit of length.

To use Equations 34-37 requires that dynamic unbalance about all three axes be known, but it is unnecessary to know any phase angles, except to define signs of w , x , y , and z . The signs of x , y , and z must define the unique location of a positive or negative w , and require a convention.

Derivation of the convention depends on the established system geometry, and the established dynamic balance phase convention.

From Equation 30 it is evident that the expressions $D_x \sin \alpha / 2xy$, $D_y \sin \beta / 2yz$ and $D_z \sin \gamma / 2zx$ must *all* yield W as either positive or negative. A positive W means correction by weight addition (W at x, y, z and W at $-x, -y, -z$). A negative W means that correction *could* consist of weight removal, i.e., removing W at x, y, z and at $-x, -y, -z$. However, this is not a practical correction means; therefore derivation of an equivalent correction by weight addition is necessary. With a three-dimensional model it can be demonstrated that

$-W, +x, +y, +z$	is equivalent to	$+W, -x, +y, +z$	or	$+W, +x, -y, +z$	or	$+W, +x, +y, -z$
$-W, -x, -y, -z$		$+W, +x, -y, -z$		$+W, -x, +y, -z$		$+W, -x, -y, +z$
		$+2W, 0, +y, -z$		$+2W, +x, 0, -z$		$+2W, +x, -y, 0$
		$+2W, 0, -y, +z$		$+2W, -x, 0, +z$		$+2W, -x, +y, 0$

It is now apparent why triaxial balancing requires either three or five weights. Static unbalance requires one weight N . Dynamic unbalance requires two weights if W from Equation 30 is positive; if W is negative the equivalent correction by weight addition requires four weights.

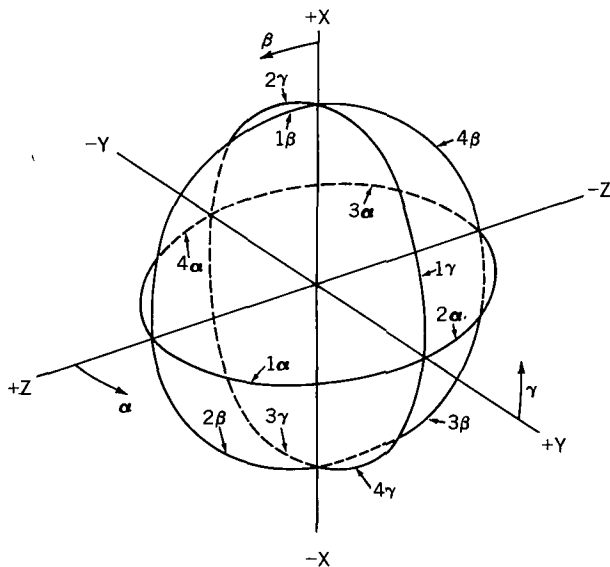


Figure 2—Isometric representation of the coordinate system.

It will be shown that the sign of W , and therefore the number of correction weights, can be predicted from any two dynamic unbalance phase quadrants.

Figure 2 represents the octants of a sphere with the signs of x, y , and z and the trigonometric quadrants of α, β , and γ . Data for each octant can be tabulated as in Table 1. By abstracting significant data from Table 1 and considering the implication of a negative W , Tables 2 and 3 may be derived.

In Table 1, A through H represent the octants of the sphere, and W must be defined in two diametrically opposed octants. Data for octants E-H apply to the $-x$ hemisphere. The second column gives the signs of x, y , and z .

Column 3 shows whether a positive W (correction by weight addition) or a negative W (correction by weight removal) is to be defined. Column 4 shows the quadrant of the phase of dynamic unbalance (about each axis); correction in the corresponding octant is implied. This is the quadrant of α, β , or γ in the octant *only* if the octant is on the *positive* side of the x, y , or z axis. Because

Table 1
Spherical Octant Correction Signs.

Octant	Coordinate x y z	Positive or Negative Correction	Unbalance Phase Quadrant a β γ	Sine a β γ	Coordinate Product xy yz zx	W		
						$\frac{D_x \sin \alpha}{2xy}$	$\frac{D_y \sin \beta}{2yz}$	$\frac{D_z \sin \gamma}{2zx}$
A	+ + +	+	1 1 1	+ + +	+ + +	+	+	+
		-	3 3 3	- - -	+ + +	-	-	-
B	+ + -	+	2 4 ③	+ - -	+ - -	+	+	+
		-	4 2 ①	- + +	+ - -	-	-	-
C	+ - -	+	3 ② ④	- + -	- + -	+	+	+
		-	1 ④ ②	+ - +	- + -	-	-	-
D	+ - +	+	4 ③ 2	- - +	- - +	+	+	+
		-	2 ① 4	+ + -	- - +	-	-	-
E	- - -	+	① ① ①	+ + +	+ + +	+	+	+
		-	③ ③ ③	- - -	+ + +	-	-	-
F	- - +	+	② ④ 3	+ - -	+ - -	+	+	+
		-	④ ② 1	- + +	+ - -	-	-	-
G	- + +	+	③ 2 4	- + -	- + -	+	+	+
		-	① 4 2	+ - +	- + -	-	-	-
H	- + -	+	④ 3 ②	- - +	- - +	+	+	+
		-	② 1 ④	+ + -	- - +	-	-	-

of the dynamic unbalance phase convention, if the octant is on the *negative* side the unbalance phase quadrant is diametrically opposed to the octant phase quadrant; such cases are noted by the circle around the quadrant number. The concept involved requires three-dimensional visualizing to define the unbalance phase quadrant in each specific case, and column 4 lists the results of this exercise.

For convenience and emphasis the dynamic unbalance phase convention will now be repeated: α is the phase of a light spot with a positive x coordinate about axis XX ; β is the phase of a light spot with a positive y coordinate about axis YY ; γ is the phase of a light spot with a positive z coordinate about axis ZZ .

Table 2

Corrections Based on Table 1.

Unbalance Phase Quadrant			Coordinates for Corrections W			W	Correction
α	β	γ	x	y	z		
1	1	1	+	+	+	+	Add as indicated (2 weights)
1	4	2	+	-	-	-	Use Table 3 (4 weights)
2	4	3	+	+	-	+	Add as indicated (2 weights)
2	1	4	+	-	+	-	Use Table 3 (4 weights)
3	2	4	+	-	-	+	Add as indicated (2 weights)
3	3	3	+	+	+	-	Use Table 3 (4 weights)
4	3	2	+	-	+	+	Add as indicated (2 weights)
4	2	1	+	+	-	-	Use Table 3 (4 weights)

Table 3

Equivalent Corrections by Weight Addition
for Negative W Cases in Table 2.

Correction Choice	Quadrant			W			W'		
	α	β	γ	x	y	z	x'	y'	z'
A*	1	4	2	+	+	+	0	+	-
	2	1	4	+	+	-	0	+	+
	3	3	3	+	-	-	0	+	-
	4	2	1	+	-	+	0	+	+
B†	1	4	2	+	+	-	+	0	+
	2	1	4	+	+	+	+	0	-
	3	3	3	+	-	+	+	0	-
	4	2	1	+	-	-	+	0	+
C‡	1	4	2	+	-	+	+	+	0
	2	1	4	+	-	-	+	+	0
	3	3	3	+	+	-	+	-	0
	4	2	1	+	+	+	+	-	0

$$* W' = 2W(y^2 + z^2), \quad |y'| = y/\sqrt{y^2 + z^2},$$

$$|z'| = z/\sqrt{y^2 + z^2}.$$

$$† W' = 2W(z^2 + x^2), \quad |x'| = x/\sqrt{z^2 + x^2},$$

$$|z'| = z/\sqrt{z^2 + x^2}.$$

$$‡ W' = 2W(x^2 + y^2), \quad |x'| = x/\sqrt{x^2 + y^2},$$

$$|y'| = y/\sqrt{x^2 + y^2}.$$

Since column 2 includes all octants, column 4 includes *all physically possible* α , β , and γ phase quadrant combinations. Octants E-H are included for completeness but it is sufficient to define one of the two weights W , and the one in the $+x$ hemisphere can be chosen. In other words x can be considered always positive, by convention.

Table 2 includes all physically possible dynamic unbalance phase quadrant combinations. There are 8 of these: 111, 243, 324, and 432, which imply correction by two weights, and 142, 214, 333, and 421, which imply correction by four weights, placed as indicated in Table 3. Note that if the quadrant of α is known, there are only two possible quadrants for β and these are always adjacent quadrants. Also, α and β determine γ since $\tan \alpha \tan \beta \tan \gamma = 1$.

Column 5 of Table 1 shows the sign of $\sin \alpha$, $\sin \beta$, and $\sin \gamma$ corresponding to the quadrant tabulation of column 4. Column 6 gives the signs of the products xy , yz , and zx based on the column 2 signs for x , y , and z . Column 7 tabulates the signs of W , determined from Equation 30, by using xy , yz , and zx from column 6 and $\sin \alpha$,

$\sin \beta$, and $\sin \gamma$ from column 5. The fact that the results uniformly conform to column 3 verifies the entire table. Note that D_x , D_y , and D_z are positive by definition.

APPLICATION OF GENERAL PRINCIPLES TO THE PROBLEM OF BALANCING A SPACECRAFT TRIAXIALLY

Expressing Triaxial Unbalance and Unbalance Tolerance

Just as single unbalance tolerance is often expressed as center of gravity displacement and principal axis tilt, triaxial unbalance tolerance basically limits center of gravity displacement (from a point), i.e., δ , and tilt of all three principal axes (θ_x , θ_y , θ_z). The values θ_x , θ_y , and θ_z are likely to be equal, or, in some cases, two of them may be equal and the third considerably smaller. Analytically, all three axes have identical significance, hence the triangular symmetry of the preceding derivations. In practice, unbalance tolerances may define a nonspherical volume limit for the center of gravity and principal axis limits within volumes other than circular cones.

In any event, in expressing static unbalance magnitude:

$$\delta = \frac{\sqrt{S_x^2 + S_y^2 + S_z^2}}{M \sqrt{2}} = \frac{NR}{M} \quad (38)$$

where R is the assigned value. Note that R , N , and M define unbalance magnitude. It is necessary to have x_N , y_N , and z_N to define the direction of unbalance.

It is pertinent to distinguish between dynamic unbalance and product of inertia. Although dynamic unbalance is a product of inertia, and product of inertia is a dynamic unbalance, there is a conceptual difference.

An unbalanced object, with reference axes XX , YY , and ZZ , has principal axes $X'X'$, $Y'Y'$, and $Z'Z'$ skewed θ_x , θ_y , and θ_z from the reference axes. Two of the principal axes are axes of maximum and minimum inertia, and the three principal axes are mutually perpendicular.

The product of inertia concept is that the mass of the body is projected onto a plane defined by two perpendicular axes, such as XX and YY . The product of inertia I_{xy} is then $\int_{xy} dm$; i.e., the integral summation of mass elements times the product of their XY coordinates. Products of inertia about pairs of principal axes are all zero. I_{xy} , I_{yz} , and I_{zx} will have positive or negative values.

The dynamic unbalance concept, for dynamic unbalance D_x/a about axis XX , is that all the mass of the body is projected onto a plane XP defined by axis XX and an axis PP in the YZ plane oriented so that the product of inertia I_{xp} is maximum.

$|I_{xp}|$ is the magnitude D_x of dynamic unbalance, and the orientation of PP in the YZ plane defines the phase angle α of dynamic unbalance. If an axis QQ is perpendicular to PP and in the YZ

plane, then I_{xQ} will be zero. Dynamic unbalance is zero about all three principal axes, and some positive value for any other axis. It is always positive because its phase angle defines vector direction—a negative value at phase angle α is stated as an equal positive value at phase angle $(\alpha + 180^\circ)$. A phase convention is essential, and so is a convention on whether "unbalance" means existing net mass moment (heavy spot) or indicated correction (light spot). In this paper, unbalance means indicated correction.

It is noteworthy that, although only the principal axes are axes of zero dynamic unbalance, there is for any axis AA one perpendicular axis BB such that the product of inertia I_{AB} is zero.

The distinction between product of inertia and dynamic unbalance has been clarified at some length because it is rarely explicitly stated. For the conventions established, it is apparent that:

$$D_x = \sqrt{I_{xy}^2 + I_{xz}^2} \tan^{-1} \frac{I_{xy}}{I_{xz}}, \text{ and } I_{xy} = D_x \sin \alpha = D_y \cos \beta, \quad (39)$$

$$D_y = \sqrt{I_{yz}^2 + I_{xy}^2} \tan^{-1} \frac{I_{yz}}{I_{xy}}, \text{ and } I_{yz} = D_y \sin \beta = D_z \cos \gamma, \quad (40)$$

$$D_z = \sqrt{I_{zx}^2 + I_{zy}^2} \tan^{-1} \frac{I_{zx}}{I_{zy}}, \text{ and } I_{zx} = D_z \sin \gamma = D_x \cos \alpha. \quad (41)$$

The principal axis tilt in the XY plane is $(1/2)\tan^{-1} (2I_{xy}/|I_{xx} - I_{yy}|)$ and, for small tilt angles (< 10 degrees), it may be considered $I_{xy}/|I_{xx} - I_{yy}|$ radians. Therefore under the assumption of a tilt angle < 10 degrees and no cosine error (cosine error would actually be minimal)

$$\theta_x \approx \sqrt{\left(\frac{I_{xy}}{I_{xx} - I_{yy}}\right)^2 + \left(\frac{I_{xz}}{I_{xx} - I_{zz}}\right)^2}. \quad (42)$$

Equation 42 may be considered exact for all practical purposes since θ_x would be less than 10 degrees. By using Equations 39 and 41 in Equation 42 and making some rearrangement,

$$\frac{\theta_x}{D_x} = \sqrt{\left(\frac{\sin \alpha}{I_{xx} - I_{yy}}\right)^2 + \left(\frac{\cos \alpha}{I_{xx} - I_{zz}}\right)^2}. \quad (43)$$

Similarly

$$\frac{\theta_y}{D_y} = \sqrt{\left(\frac{\sin \beta}{I_{yy} - I_{zz}}\right)^2 + \left(\frac{\cos \beta}{I_{yy} - I_{xx}}\right)^2}, \quad (44)$$

$$\frac{\theta_z}{D_z} = \sqrt{\left(\frac{\sin \gamma}{I_{zz} - I_{xx}}\right)^2 + \left(\frac{\cos \gamma}{I_{zz} - I_{yy}}\right)^2}. \quad (45)$$

Equations 43-45 relate the tilt of each principal axis from an adjacent reference axis to the corresponding dynamic unbalance about the reference axis. That is, with I_{xx} , I_{yy} , I_{zz} , and D_x/α known θ_x can be calculated and compared to tolerance, as can θ_y , with D_y/β known, or θ_z , with D_z/γ known. How to establish limits for D_x , D_y , and D_z from a knowledge of θ_x , θ_y , and θ_z is less apparent, but can be done as follows: Study of a general function $\sin^2 e/A + \cos^2 e/B$ shows that the maximum limit for D_x in Equation 43 would occur when either $\sin \alpha = 0$ and $\cos \alpha = 1$, or $\sin \alpha = 1$ and $\cos \alpha = 0$. It follows that the limits for D_x , D_y , and D_z are:

$$D_x \leq (I_{xx} - I_*) \theta_x, \quad (46)$$

$$D_y \leq (I_{yy} - I_*) \theta_y, \quad (47)$$

$$D_z \leq (I_{zz} - I_*) \theta_z. \quad (48)$$

I_* must, in each case, be chosen from I_{xx} , I_{yy} , and I_{zz} to make D_x , D_y , or D_z a minimum but not zero. Note that these equations are predicated on the most unfavorable phase orientations of α , β , and γ . Larger D_x , D_y , and D_z may or may not be compatible with θ_x , θ_y , and θ_z limits, and more than 41.4 percent larger would never be compatible.

Triaxial dynamic unbalance could be expressed as D_x/α , D_y/β , D_z/γ . This fully defines all parameters, and principal axes tilts can be determined if I_{xx} , I_{yy} , and I_{zz} are known, but there is no indication of the necessary corrections. If separate corrections were applied for D_x , D_y , and D_z , six weights would be needed, weight addition would be excessive, and the result would be *that dynamic balance would not exist about xx, yy, or zz*, because of interaction between the corrections. Also D_x/α , D_y/β , D_z/γ contains redundant data— D_x/α and D_y/β fully define the unbalance.

A better way to express triaxial unbalance, both static and dynamic, is by N , x_N , y_N , and z_N and W , x , y , and z . This defines the situation and also the necessary correction. Equations 23-29 enable derivation of D_x/α , D_y/β , and D_z/γ from W , x , y , and z . It is to be understood that N and W are on the surface of a sphere of unit radius (and therefore x_N , y_N , z_N , x , y , and z will all be less than one) and that correction W is needed at $+x$, $+y$, $+z$, and also at $-x$, $-y$, $-z$. N is directly indicative of the magnitude of triaxial static unbalance (x_N , y_N , z_N define its direction) and W is indicative of the magnitude of triaxial dynamic unbalance, with x , y , z defining its direction; x_N , y_N , z_N and x , y , z are direction cosines of the unbalance vectors.

It is noteworthy that, although $N = 0.707 \sqrt{S_x^2 + S_y^2 + S_z^2}$, W is neither equal to $0.707 \sqrt{D_x^2 + D_y^2 + D_z^2}$ nor related to it in any simple fashion. W is actually

$$\sqrt{\frac{D_x^2 + D_y^2 + D_z^2}{8(x^2y^2 + y^2z^2 + z^2x^2)}}$$

where $x^2 + y^2 + z^2 = 1$.

Computing the Effect of Changes to a Spacecraft

Suppose the balance condition of a spacecraft is $N, x_N, y_N, z_N, W, x, y, z$, and a weight C is added at p, q, r . The new balance condition can be defined as follows below: (For convenience the symbols d, e, f will be used in place of x_N, y_N , and z_N and $N', d', e', f', W', x', y', z'$ will mean the new balance condition.)

Note that W, x, y, z includes all dynamic balance effects of N, d, e, f , and since W, x, y, z implies $W, +x, +y, +z$, plus $W, -x, -y, -z$, it has no static balance effect. For consistency, addition of C at $+p, +q, +r$ can be coded as $+C, -p, -q, -r, -C, +p, +q, +r$, which is actually the correction needed by the unbalance due to C (i.e., add C at $-p, -q, -r$ and then deduct C at $+p, +q, +r$ and at $-p, -q, -r$).

The problem can be stated as follows:

$$(N, d, e, f, W, x, y, z) \text{ (known input)} \oplus (+C, -p, -q, -r, -C, +p, +q, +r) = (N', d', e', f', W', x', y', z',)$$

meaning two known unbalances are to be superposed to give a third in the same form of expression as the first two. The problem is how to perform the superposition operation \oplus . The information content of N, d, e, f, W, x, y, z is

$$\text{INPUT A} = \left\{ \begin{array}{l} (S_X / \underline{a})_A = N \sqrt{e^2 + f^2} \angle \tan^{-1} \frac{e}{f} , \\ *(S_Y / \underline{b})_A = N \sqrt{f^2 + d^2} \angle \tan^{-1} \frac{f}{d} , \\ (S_Z / \underline{c})_A = N \sqrt{d^2 + e^2} \angle \tan^{-1} \frac{d}{e} , \\ *(D_X / \underline{\alpha})_A = 2Wx \sqrt{y^2 + z^2} \angle \tan^{-1} \frac{y}{z} , \\ *(D_Y / \underline{\beta})_A = 2Wy \sqrt{z^2 + x^2} \angle \tan^{-1} \frac{z}{x} , \\ (D_Z / \underline{\gamma})_A = 2Wz \sqrt{x^2 + y^2} \angle \tan^{-1} \frac{x}{y} , \\ d^2 + e^2 + f^2 = x^2 + y^2 + z^2 = R_A^2 = 1 \text{ (by definition)} . \end{array} \right.$$

Note that, since $R = 1$ by convention, the information content of N, d, e, f, W, x, y, z could be represented by N, d, e, W, x, y . The information content of $+C, -p, -q, -r, -C, p, q, r$ is

$$\text{INPUT B} = \left\{ \begin{array}{l} * (S_X/\underline{a})_B = +C\sqrt{q^2 + r^2} / \tan^{-1} \frac{-q}{-r} \text{ and } (D_X/\underline{a})_B = 2p(S_X/\underline{a})_B \\ * (S_Y/\underline{b})_B = +C\sqrt{r^2 + p^2} / \tan^{-1} \frac{-r}{-p} \text{ and } (D_Y/\underline{b})_B = 2q(S_Y/\underline{b})_B \\ (S_Z/\underline{c})_B = +C\sqrt{p^2 + q^2} / \tan^{-1} \frac{-p}{-q} \text{ and } (D_Z/\underline{c})_B = 2r(S_Z/\underline{c})_B \\ p^2 + q^2 + r^2 = R_B^2 \text{ (which may have any value).} \end{array} \right.$$

Input B could be coded as $-C, p, q, r$. Note that for input B, $a = \alpha, b = \beta, c = \gamma$, and the value of R_B does not affect the fact that $(S_X/\underline{a})_B, (S_Y/\underline{b})_B, (D_X/\underline{a})_B$, etc. are the corrections for $+C$ at $+p, +q, +r$; i.e., $-C$ at $+p, +q, +r$. It has been shown earlier that $-C, p, q, r$ may actually imply five individual weight additions.

Some of the information content of inputs A and B is not independent and therefore redundant, and only the items marked with an asterisk are needed to perform the operation input A \oplus input B. This involves vector addition of corresponding elements of the inputs, as follows:

$$R = 1 \text{ (by definition) ,}$$

$$(S_X/\underline{a})_A \rightarrow (S_X/\underline{a})_B = (S_X/\underline{a})_C ,$$

$$(S_Y/\underline{b})_A \rightarrow (S_Y/\underline{b})_B = (S_Y/\underline{b})_C ,$$

$$(D_X/\underline{a})_A \rightarrow (D_X/\underline{a})_B = (D_X/\underline{a})_C ,$$

$$(D_Y/\underline{b})_A \rightarrow (D_Y/\underline{b})_B = (D_Y/\underline{b})_C .$$

This is sufficient information to derive *output C* in the form $N', d', e', f', W', x', y', z'$ (redundant components are not included). It has already been established that $(S_X/\underline{a})_C$ and $(S_Y/\underline{b})_C$ will yield N', d', e', f' for output C (Equations 9-16). Also $(D_X/\underline{a})_C$ and $(D_Y/\underline{b})_C$ will yield W', x', y', z' for output C (Equations 30-37). Therefore output C is obtained in the form $N', d', e', f', W', x', y', z'$, or can be coded as N', d', e', W', x', y' (since $R = 1$).

The \oplus operation is quite tedious by manual methods, but would be trivial for a suitably programmed computer, which could readily evaluate the balance condition of a complicated assembly from any number of inputs coded in the form $-C, p, q, r$. It could also evaluate total spacecraft weight (ΣC) and inertias about reference axes XX, YY , and ZZ , which are $\Sigma C(p^2 + q^2)$, $\Sigma C(q^2 + r^2)$, and $\Sigma C(r^2 + p^2)$ except that the inertias would not include the body-centered inertias of components. The dynamic unbalance results would also fail to include the body-centered dynamic unbalance of components.

These body-centered mass parameters could be included in a computer program with little additional complication, and so could inputs coded in polar rather than rectangular coordinates.

Prerequisite Revision of Measured Unbalance Data

To solve for either static or dynamic correction it is necessary to compensate for the fact that the input data is interdependent, partially redundant, and to some degree inaccurate, and therefore inconsistent. The inconsistency comes from measurement inaccuracies and also from the fact that the axes which are *theoretically* orthogonal and intersecting are neither *perfectly* orthogonal nor *exactly* intersecting.

To obtain an optimized solution by the methods to be described requires first that the input data be "normalized." Normalizing makes all the input data compatible and also averages out random errors. All raw data is assumed equally reliable, and for vectors this implies that the angular error in degrees is approximately equal numerically to 0.6 the magnitude error expressed as a percentage (for example, a 2 percent error in magnitude is approximately equivalent to a 1 degree error in phase angle.)

It has been shown that only certain combinations of unbalance phase angle quadrants are physically possible. It can also be derived that for static unbalances

$$S_X \cos a = S_Y \sin b ,$$

$$S_Y \cos b = S_Z \sin c ,$$

$$S_Z \cos c = S_X \sin a ,$$

as given in Equations 17-19, and for dynamic unbalances

$$D_X \cos \alpha = D_Z \sin \gamma , \quad (49)$$

$$D_Y \cos \beta = D_X \sin \alpha , \quad (50)$$

$$D_Z \cos \gamma = D_Y \sin \beta . \quad (51)$$

These relationships would hold exactly for perfect data, but are met only imperfectly by empirical measurements. Therefore, normalizing data means adjusting it to conform to Equations 17-19 and 49-51. This is best illustrated graphically, but an equivalent algebraic method can easily be derived.

Figure 3 shows *exact* unbalance vectors $S_X \angle a$ (OX), $S_Y \angle b$ (OY), and $S_Z \angle c$ (OZ) on a common polar plot. The rest of the diagram illustrates that these vectors conform to Equations 17-19.

Figure 4 shows, on a similar plot, measured values OX' , OY' , and OZ' of unbalance about the three axes. The rest of the diagram shows their nonconformity with the equations. The rectilinear

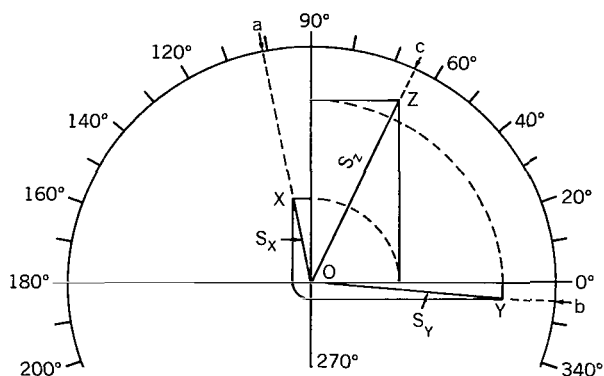


Figure 3—Plot of exact unbalance vectors.

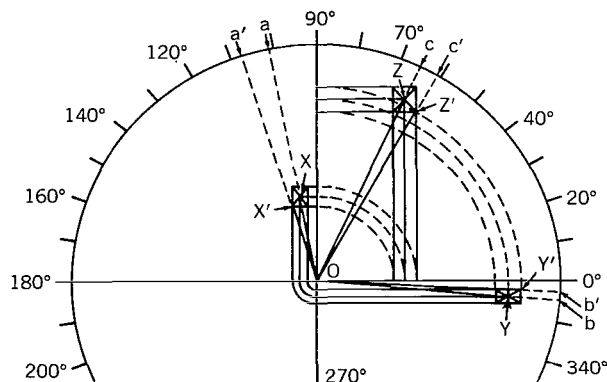


Figure 4—Normalizing plot for inexact unbalance vectors.

projection of rotated components of the original vectors defines three rectangular areas, each having X' , Y' , or Z' as one corner, and X , Y , or Z as geometric center.

OX , OY , and OZ satisfy Equations 17-19 and are the normalized values of the original vectors OX' , OY' , and OZ' . All three original vectors are changed in both magnitude and phase. The normalized vectors are not necessarily exact, though they tend to define the overall condition better than the original measurements, but they are constrained to be consistent and therefore the calculation of unbalance correction is not complicated by multiple results based on the choice of input parameters from redundant data. Also, inaccurate measured data could appear to define a physically impossible situation (imaginary solutions of equations), whereas normalized data cannot.

The procedure for normalizing static unbalance data has been described. The procedure for dynamic unbalance data is the same, except that Equations 49-51 apply. In all cases measured data may be expected to conform approximately with the applicable equations. Extreme deviation would imply a major mistake rather than normal inaccuracy. With $S_x \underline{a}$, $S_y \underline{b}$, $S_z \underline{c}$, $D_x \underline{a}$, $D_y \underline{b}$, $D_z \underline{c}$ considered to be normalized input data unbalance correction calculations can be made by using Equations 13-19 and 34-37.

Computing Dynamic Unbalance Corrections

An essential operation in determining dynamic unbalance correction is deriving a basic correction W , x , y , z from normalized data $D_x \underline{a}$, $D_y \underline{b}$, $D_z \underline{c}$. The symbol $\triangle_{\alpha\beta\gamma}^{xyz}$ will be used to denote this operation, which consists of solving Equations 34-37 for W , x , y , and z . In these equations x , y , and z can have real values only if the sum of any two of the quantities D_x^2 , D_y^2 , and D_z^2 exceeds the third quantity. This condition will be met by exact or normalized values, but may not be satisfied by measured values.

It is necessary to use previously derived tabulations to determine the signs of x , y , z , and W based on the quadrants of unbalance phase angles. Table 4, which is taken from Table 2, covers all cases for which W is positive.

Table 4
Coordinate Signs for Dynamic
Correction with Two Weights.

Phase Quadrant			Sign			
α	β	γ	W	x	y	z
1	1	1	+	+	+	+
2	4	3	+	+	+	-
3	2	4	+	+	-	-
4	3	2	+	+	-	+

Although there is a possibility of these positive W cases occurring, a real spacecraft configuration would very often need the equivalent of a negative W correction, with +W at x, y, z, and +W' at x', y', z' (Table 3).

Table 3 indicates the correction required at the surface of a sphere of unit radius, and the preferable choice between A, B, and C would be that for which W' is a minimum; i.e., A if y and z are both less than x, B if x and z are both less than y, and C if x and y are both less than z.

The tables give the coordinates of correction weights in the positive x hemisphere; it is to be understood that equal but diametrically opposed weights are also implied in the negative x hemisphere (i.e., reverse signs of x, y, z, x', y', and z').

$\triangle_{\alpha\beta\gamma}^{xyz}$ is the computation of W, x, y, and z, and W', x', y', z' in Table 3, or of W, x, y, z in Table 4, whichever is applicable to the phase quadrants of α , β , and γ . $\triangle_{\alpha\beta\gamma}^{xyz}$ will yield a triaxial dynamic unbalance correction; however alternative corrections may be preferable because they involve less added weight, because weight addition may be more convenient, or because the accuracy of correction may be less dependent on precise placement of weights. Advantages of minimum weight and minimum dependence on weight placement accuracy will, in general, be concurrent and associated with maximum angular displacement of correction weights from all of the reference axes.

Alternative corrections are obtained by adding a triaxially symmetric unbalance vector and correcting for the modified unbalance situation.

A triaxially symmetric vector means an opposed pair of weights located so that they provide equal unbalance effect about all of the reference axes. There are only four possible locations with positive x coordinates on the "unit sphere" surface:

1. x = +0.578, y = +0.578, z = +0.578 ($\alpha = 45^\circ$, $\beta = 45^\circ$, $\gamma = 45^\circ$),
2. x = +0.578, y = +0.578, z = -0.578 ($\alpha = 135^\circ$, $\beta = 315^\circ$, $\gamma = 225^\circ$),
3. x = +0.578, y = -0.578, z = -0.578 ($\alpha = 225^\circ$, $\beta = 135^\circ$, $\gamma = 315^\circ$),
4. x = +0.578, y = -0.578, z = +0.578 ($\alpha = 315^\circ$, $\beta = 225^\circ$, $\gamma = 135^\circ$).

A weight W at one of these locations (plus an equal weight diametrically opposed on the "negative x" hemisphere) will cause $D_x = D_y = D_z = 0.942W = D_p$ with phase angles α , β , γ as indicated (0.578 is an approximation of $1/\sqrt{3}$ and 0.942 of $(2/3)\sqrt{2}$).

Figure 5 is a common plot of normalized initial unbalance vectors $D_x \angle (OX)$, $D_y \angle (OY)$, and $D_z \angle (OZ)$. $D_p(OP_{123})$ represents a triaxially symmetric vector, per case 1 above, $D_x' \angle (P_{123}X)$, $D_y' \angle (P_{123}Y)$, and $D_z' \angle (P_{123}Z)$ represent the correction needed after applying D_p . Although D_p must be at 45 degrees, its magnitude is arbitrary and therefore an infinite number of possible

solutions exists. In all cases D_X'/α' , D_Y'/β' , D_Z'/γ' will be normalized, i.e., consistent, and the total ballast weight required will be a function of the magnitude of D_P : total weight = $(3/\sqrt{2})D_P$ plus the weight indicated by $\Delta_{\alpha'\beta'\gamma'}$.

The total weight can be expressed in terms of D_P and known data, and solving for zero values of the derivative of this expression would yield the optimum value of D_P for minimum weight correction. However, this would be a prohibitively cumbersome operation since the total weight is

$$W_{TOTAL} = \frac{3}{\sqrt{2}} D_P + 2 \left(K_1 + K_2 + \frac{1}{2} \right) \sqrt{\frac{D_P^2 + D_Z^2 - \sqrt{2} D_P D_Z (\cos \gamma + \sin \gamma)}{K_Z (K_X + K_Y)}}, \quad (52)$$

where

$$K_X = \frac{1}{1 + \left(\frac{-x^2 + y^2 + z^2}{+x^2 - y^2 + z^2} \right) + \left(\frac{-x^2 + y^2 + z^2}{+x^2 + y^2 - z^2} \right)},$$

$$K_Y = \frac{1}{1 + \left(\frac{+x^2 - y^2 + z^2}{-x^2 + y^2 + z^2} \right) + \left(\frac{+x^2 - y^2 + z^2}{+x^2 + y^2 - z^2} \right)},$$

$$K_Z = \frac{1}{1 + \left(\frac{+x^2 + y^2 - z^2}{-x^2 + y^2 + z^2} \right) + \left(\frac{+x^2 + y^2 - z^2}{+x^2 - y^2 + z^2} \right)},$$

(note that $K_X^2 + K_Y^2 + K_Z^2 = 1$). K_1 and K_2 are the smaller two of K_X , K_Y , and K_Z if the phase quadrant combination of α' , β' , γ' is 142, 214, 333, or 421, but are both zero if the phase quadrant combination of α' , β' , γ' is 111, 243, 324, or 432;

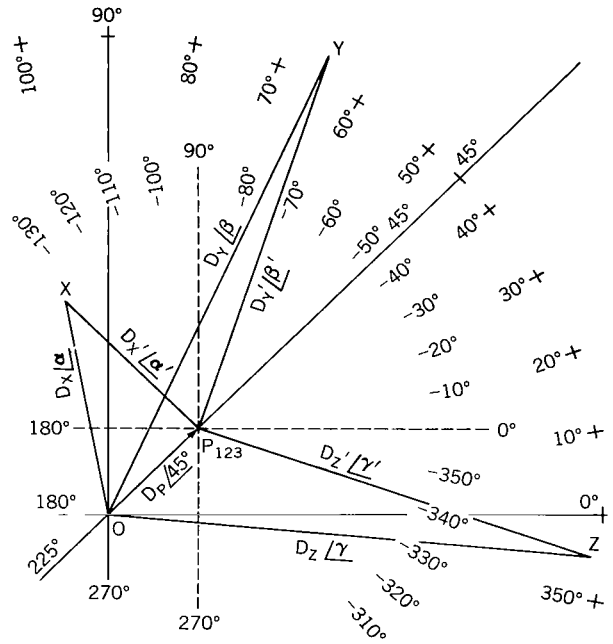


Figure 5—Vector plot for case 1 of a triaxially symmetric vector D_P .

$$x^2 = D_P^2 + D_X^2 - \sqrt{2} D_P D_X (\cos \alpha + \sin \alpha),$$

$$y^2 = D_P^2 + D_Y^2 - \sqrt{2} D_P D_Y (\cos \beta + \sin \beta),$$

$$z^2 = D_P^2 + D_Z^2 - \sqrt{2} D_P D_Z (\cos \gamma + \sin \gamma),$$

$$\alpha' = \tan^{-1} \frac{D_X \sin \alpha - 0.707 D_P}{D_X \cos \alpha - 0.707 D_P},$$

$$\beta' = \tan^{-1} \frac{D_Y \sin \beta - 0.707 D_P}{D_Y \cos \beta - 0.707 D_P},$$

$$\gamma' = \tan^{-1} \frac{D_z \sin \gamma - 0.707 D_p}{D_z \cos \gamma - 0.707 D_p}.$$

Equation 52 holds only for case 1 of the triaxially symmetric vector D_p . It would have to be modified by geometrically appropriate changes of the signs of some of the trigonometric functions to be valid for cases 2, 3, or 4.

The total weight function was evaluated for specific values of $D_x/\underline{\alpha}$, $D_y/\underline{\beta}$, and $D_z/\underline{\gamma}$ and various case 1 values of D_p . Setting $D_p = 0$ gave the minimum number of correction weights, but not the minimum total weight. The total weight function had more than one minimum and at least one relative maximum. It became infinite when α' , β' , or γ' was 0° , 90° , 180° , or 270° . Further evaluation of the function could be made by expressing its derivative analytically, a formidable task, or by curve plotting based on iterative computation for incremental values of D_p . The latter would be tedious but relatively straightforward. If D_x were larger than D_y and D_z , D_p could be limited to the value of D_x and increased in increments of $0.01 D_x$ from zero to D_x .

Figure 5 illustrates only case 1 of the four possible triaxially symmetric unbalance vectors. Each of the other three cases is represented on a common vector plot by equal vectors at 135° , 225° , and 315° , as shown in Figure 6, where OP_1 is the contribution to D_x , OP_2 the contribution to D_y , and OP_3 the contribution to D_z .

Figure 7 shows initial unbalances OX , OY , OZ with a case 2 triaxially symmetric vector represented by OP_1 , OP_2 , and OP_3 , and a case 3 symmetric vector represented by OP_1' , OP_2' , and OP_3' . Note that P_1 and P_1' are joined to X , P_2 and P_2' to Y , and P_3 and P_3' to Z .

Figure 8 shows a similar plot for a case 4 symmetric vector. For all cases, the magnitude of the symmetric vector is arbitrary (OP_1 , OP_2 , and OP_3 must be equal but can have any length) and therefore each case represents a range of possible solutions. Also $D_x'/\underline{\alpha'}$, $D_y'/\underline{\beta'}$, and $D_z'/\underline{\gamma'}$ will be normalized, and the total ballast weight needed will be $(3/\sqrt{2}) D_p$ plus the weight indicated by $\triangle_{\alpha'\beta'\gamma'}$.

To consider all possibilities it would be necessary to evaluate all the cases for all values of D_p . However, some cases can be eliminated by inspection. If α, β, γ are all in the first quadrant, only case 1 need be considered, and if α, β, γ are all in the third quadrant (see Figure 8), only

cases 2, 3, and 4 need be considered. Apart from these two general rules, a vector plot may allow some elimination. For example, case 3 could be eliminated in the situation shown by Figure 7, because a case 3 triaxially symmetric vector opposes all three of the original unbalance vectors. Of course, computer programming could be used to evaluate all possibilities and this might be better than including selective rejection rules in the program.

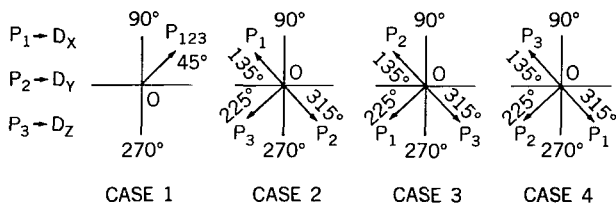


Figure 6—Common vector plots of the four possible triaxially symmetric vector cases.

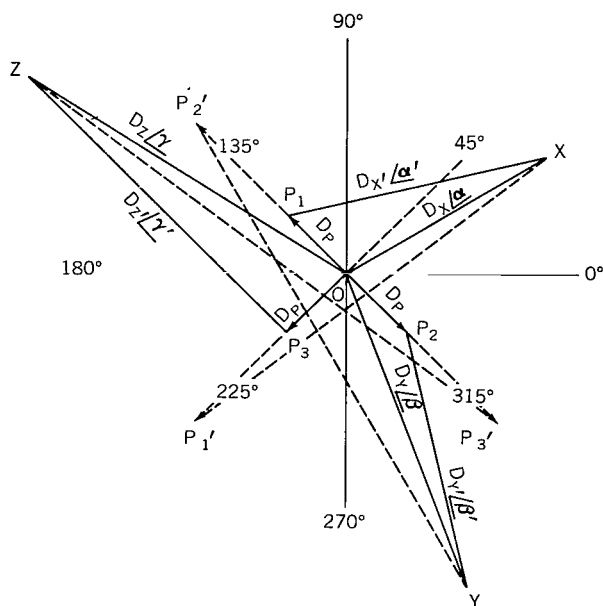


Figure 7—Vector plots for case 2 (—) and case 3 (---) of triaxially symmetric vectors.

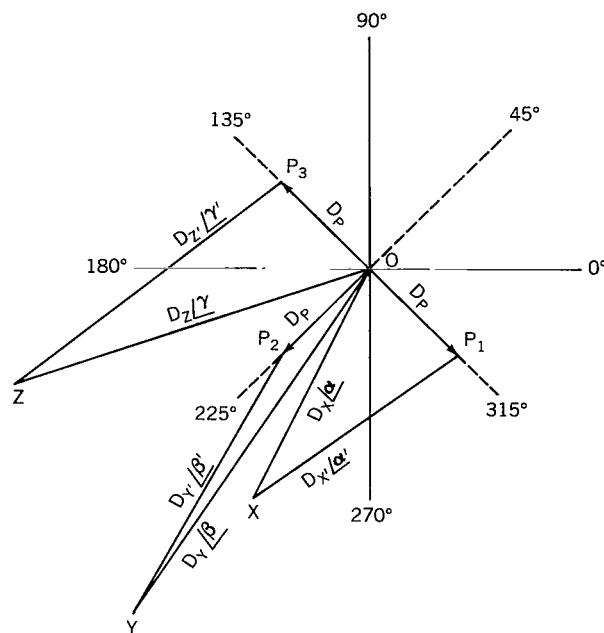


Figure 8—Vector plot for case 4 of a triaxially symmetric vector D_p .

In summary, computation of the optimum dynamic unbalance correction consists of assuming ranges of values for each of the four symmetric vectors, evaluating the weight needed for each symmetric vector plus the weight needed for correction of the remaining unbalance, and selecting the result which yields the minimum total weight to be added.

At this point it is pertinent to consider the advantages of finding an optimum (minimum weight) solution rather than, for instance, that given by $\triangle_{a\beta\gamma}^{xyz}$ with no symmetric vector.

The obvious advantage is saving weight, and the weight saved may be considerable. For a specific case (Appendix B) $\triangle_{a\beta\gamma}^{xyz}$ yielded 1886 grams, whereas an optimized solution necessitated only 1192 grams, and this was for an 83 kilogram spacecraft where the initial unbalance was comparatively small. Another advantage is that an optimized correction will, in practice, balance more precisely (assuming a "one shot" correction). The reason is that the D_p vector weight is located for minimum sensitivity to placement error and also induces the secondary corrections into locations of reduced sensitivity to placement error, i.e., it pulls them away from the reference axes. A given radial placement error has proportionally less effect at increased radius. Also, the probability of accurate placement is better for smaller weights. Advance provision for the eight possible D_p vector weights could well be advantageous.

Further justification for optimizing might be that it could easily be done by computer, and the incidental computation of many alternative corrections would, for an actual spacecraft, greatly assist in selecting an expedient correction.

For "single axis" balancing, weight optimizing, by judicious selection of correction planes, can be intuitive and alternative corrections can readily be devised if interference problems arise. But intuition is inadequate for the greater complexity of triaxial balancing, and manual computation of alternative corrections would be intolerably time consuming.

Correcting Unbalance

Unbalance can be corrected either by design or by ballasting. Balancing by design means placing or relocating components of an assembly so that the unbalance is within specified limits. It might also include such steps as changing the primary reference axes, or repositioning control elements, to conform to the actual measured or computed situation. Most of this study is quite pertinent to the general problem of designing spacecraft to be adequately well balanced in the first place so that no correction is needed. Balance is always recognized as a design factor, and development of more comprehensive and sophisticated techniques for balancing by design might well meet the needs of large spacecraft with relatively liberal unbalance tolerances. Key factors would be comprehensive and systematic mass measurement of all components, good quality control of assembly, effective error analysis, and suitable data processing and computer programs.

Correction by ballasting means adding weight in locations defined by the expression of unbalance. These locations are much more restrictive for triaxial balancing than for single axis balancing. Each of the weights needed must lie on a specific radiant line from the reference axis origin. "Splitting" weights to give the same vector effect is not impossible, but would need complicated analysis, especially for dynamic unbalance corrections.

A static correction equivalent to weight N at unit radius requires a weight of N/r at radius r .

The diametrically opposed radiants for dynamic unbalance correction are defined by $+x$, $+y$, $+z$ and $-x$, $-y$, $-z$. If at unit radius each correction weight would be W , and both weights are to be at the same radius P , each would have to be W/P^2 . If one weight L_1 is to be at radius r_1 , and the opposed weight L_2 at radius r_2 then:

$$L_1 r_1 = L_2 r_2 \quad (53)$$

Static balance must be preserved. Dynamic balance about axis XX must also be preserved. Let the included angle between the XX axis and the dynamic correction radiant be ϕ ; then

$$L_1 r_1^2 \cos \phi \sin \phi + L_2 r_2^2 \cos \phi \sin \phi = 2W(1)^2 \cos \phi \sin \phi \quad (54)$$

As $\cos \phi \sin \phi$ cancels out, other axes need not be considered. From Equations 53 and 54 it follows that

$$r_2 = \frac{2W}{L_1 r_1} - r_1 \quad (55)$$

$$L_1 = \frac{2W}{r_1 (r_1 + r_2)} \quad (56)$$

If values are assumed for W , r_1 , and L_1 , Equations 53 and 55 will give r^2 and L_2 . Note from Equation 55 that $L_1 r_1^2$ must be less than $2W$; otherwise L_2 would have to be infinite for static balance to be maintained, since for $L_1 r_1^2 = 2W$, r_2 becomes zero. If values are given to W , r_1 and r_2 Equations 53 and 56 will give L_1 and L_2 . In practice all radii should be given maximum values consistent with the spacecraft configuration, to minimize the ballast, and any effect of static balance correction on dynamic balance must be considered.

CONCLUSIONS

This study reports the derivation of analytical relationships, coordinate system coding of unbalance corrections, and various related concepts. Its intention was to promote understanding and suggest effective practical applications. Defining the N , d , e , W , x , y , or $-C$, p , q , r unbalance expression system and operation \oplus were major objectives. Its practical motivation was the needs of a particular spacecraft program (Appendix B).

Triaxial balancing is inherently more complicated than single axis balancing. However, balance tolerances are likely to be less stringent. Five problem areas are apparent:

1. How to make sufficient physical measurements to define initial unbalance. This study has defined what measurements are necessary, unless prior knowledge or acceptable estimates exist. These may be total weight, inertias about three axes, magnitude and phase of static and dynamic unbalance about one axis, and phase only of static and dynamic unbalance about a second axis. However, the effect of static unbalance correction on dynamic unbalances must be considered. This can be done by coding unbalances about $x = y = z = 0$ as $(N, d, e, W, x, y)_A$ and $(N, d, e, -N, d, e)_B$ (due to static unbalance) and performing $(N, d, e, W, x, y)_A \oplus (N, d, e, -N, d, e)_B$, or by applying static unbalance correction *before* determining dynamic unbalances. The relative merits and the best sequence of operations for either case depend on specific circumstances. There are other combinations of sufficient measurements, which suggests study of which measurements are easier to obtain with requisite accuracy. Redundant data will, if properly processed, improve the overall accuracy of balancing.

2. Relating measurements to unbalance tolerances. This study covers how it may be done.

3. Determining required corrections. This study proposes a system for expressing triaxial unbalance which implicitly defines corrections, and discusses latitude in actual weight location. The final choice depends on the specific spacecraft configuration.

4. Applying corrections. Because choice of location is very limited, this is likely to be more of a problem than in single axis balancing. Also, vector splitting techniques are difficult to evaluate, and should be considered with extreme caution, although the technique for computing the

effect of changes to spacecraft can be used to check whether two proposed weights are in fact equivalent to an indicated single weight. Computer programming would be almost essential to apply this technique very extensively.

5. Evaluating changes to spacecraft, including addition of appendages, etc., which cannot be balanced on the spacecraft. This study covers how this may be done. Again, computer programming is extremely advantageous.

(Manuscript received July 18, 1963)

Appendix A

Solution of the Triaxial Balance Equations

If it is given that

$$D_x = a = 2Wx \sqrt{y^2 + z^2} , \quad (A1)$$

$$D_y = b = 2Wy \sqrt{x^2 + z^2} , \quad (A2)$$

$$D_z = c = 2Wz \sqrt{x^2 + y^2} , \quad (A3)$$

$$x^2 + y^2 + z^2 = 1 , \quad (A4)$$

it is obvious that

$$\frac{a}{2W} = x \sqrt{y^2 + z^2} , \quad (A5)$$

$$\frac{b}{2W} = y \sqrt{x^2 + z^2} , \quad (A6)$$

$$\frac{c}{2W} = z \sqrt{x^2 + y^2} . \quad (A7)$$

Let

$$\left(\frac{a}{2W}\right)^2 = x^2(y^2 + z^2) = (xy)^2 + (xz)^2 = L , \quad (A8)$$

$$\left(\frac{b}{2W}\right)^2 = y^2(x^2 + z^2) = (xy)^2 + (yz)^2 = M , \quad (A9)$$

$$\left(\frac{c}{2W}\right)^2 = z^2(x^2 + y^2) = (xz)^2 + (yz)^2 = N . \quad (A10)$$

Also set

$$(xy)^2 = P ,$$

$$(xz)^2 = Q ,$$

$$(yz)^2 = R .$$

Therefore

$$P + Q = L , \quad (A11)$$

$$P + R = M , \quad (A12)$$

$$Q + R = N , \quad (A13)$$

and so

$$L + M = 2P + Q + R , \quad (A14)$$

From Equations A1-A3 it can be seen that

$$P = L - Q , \quad (A15)$$

$$R = M - P , \quad (A16)$$

$$Q = N - R . \quad (A17)$$

Then from Equations A14, A16, and A17

$$\begin{aligned} 2P + Q + R &= 2P + N - R + M - P \\ &= 2P + N - M + P + M - P \\ &= 2P + N . \end{aligned}$$

So $2P + N = L + M$; therefore

$$P = \frac{L + M - N}{2} . \quad (A18)$$

By reverting to original symbols:

$$\begin{aligned} (xy)^2 &= \frac{\left(\frac{a}{2W}\right)^2 + \left(\frac{b}{2W}\right)^2 - \left(\frac{c}{2W}\right)^2}{2} \\ &= \frac{a^2 + b^2 - c^2}{2(2W)^2} \\ &= \frac{a^2 + b^2 - c^2}{8W^2} . \end{aligned} \quad (A19)$$

From Equations A11 and A13

$$\begin{aligned}
L + N &= 2Q + P + R \\
&= 2Q + L - Q + M - P \\
&= 2Q + L - Q + M - L + Q \\
&= 2Q + M
\end{aligned}$$

so

$$Q = \frac{L + N - M}{2} \quad (\text{A20})$$

Again by reverting to original symbols:

$$\begin{aligned}
(xz)^2 &= \frac{\left(\frac{a}{2W}\right)^2 + \left(\frac{c}{2W}\right)^2 - \left(\frac{b}{2W}\right)^2}{2} \\
&= \frac{a^2 + c^2 - b^2}{2(2W)^2} \\
&= \frac{a^2 - b^2 + c^2}{8W^2} .
\end{aligned} \quad (\text{A21})$$

Now, from Equations A12 and A13

$$\begin{aligned}
M + N &= 2R + P + Q \\
&= 2R + L - Q + N - R \\
&= 2R + L - N + R + N - R \\
&= 2R + L ,
\end{aligned}$$

so

$$R = \frac{M + N - L}{2} \quad (\text{A22})$$

Again, in reverting to original symbols:

$$\begin{aligned}
(yz)^2 &= \frac{\left(\frac{b}{2W}\right)^2 + \left(\frac{c}{2W}\right)^2 - \left(\frac{a}{2W}\right)^2}{2} \\
&= \frac{-a^2 + b^2 + c^2}{8W^2} .
\end{aligned} \quad (\text{A23})$$

To obtain x^2 , y^2 , and z^2 , begin by dividing Equation A19 by Equation A23,

$$\begin{aligned}\frac{(xy)^2}{(yz)^2} &= \frac{\frac{a^2 + b^2 - c^2}{8W^2}}{\frac{-a^2 + b^2 + c^2}{8W^2}} , \\ \frac{x^2}{z^2} &= \frac{a^2 + b^2 - c^2}{-a^2 + b^2 + c^2} , \\ x^2 &= z^2 \left(\frac{a^2 + b^2 - c^2}{-a^2 + b^2 + c^2} \right) .\end{aligned}\tag{A24}$$

The value for y^2 can be obtained from Equations A21 and A23,

$$\begin{aligned}\frac{(xy)^2}{(xz)^2} &= \frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} , \\ y^2 &= z^2 \left(\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} \right) .\end{aligned}\tag{A25}$$

Likewise, from Equations A19 and A23

$$\begin{aligned}\frac{(yz)^2}{(xy)^2} &= \frac{-a^2 + b^2 + c^2}{a^2 + b^2 - c^2} , \\ z^2 &= x^2 \left(\frac{-a^2 + b^2 + c^2}{a^2 + b^2 - c^2} \right) .\end{aligned}\tag{A26}$$

The procedure for deriving x^2 , y^2 , and z^2 can be used to derive the following:

$$\begin{aligned}\frac{x^2}{y^2} &= \frac{(xz)^2}{(yz)^2} = \frac{a^2 - b^2 + c^2}{-a^2 + b^2 + c^2} , \\ x^2 &= y^2 \left(\frac{a^2 - b^2 + c^2}{-a^2 + b^2 + c^2} \right) = z^2 \left(\frac{a^2 + b^2 - c^2}{-a^2 + b^2 + c^2} \right) ;\end{aligned}\tag{A27}$$

$$\begin{aligned}\frac{y^2}{x^2} &= \frac{(yz)^2}{(xz)^2} = \frac{-a^2 + b^2 + c^2}{a^2 - b^2 + c^2} , \\ y^2 &= x^2 \left(\frac{-a^2 + b^2 + c^2}{a^2 - b^2 + c^2} \right) = z^2 \left(\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} \right) ;\end{aligned}\tag{A28}$$

$$\frac{z^2}{y^2} = \frac{(xz)^2}{(xy)^2} = \frac{a^2 - b^2 + c^2}{a^2 + b^2 - c^2} ,$$

$$z^2 = y^2 \left(\frac{a^2 - b^2 + c^2}{a^2 + b^2 - c^2} \right) = x^2 \left(\frac{-a^2 + b^2 + c^2}{a^2 + b^2 - c^2} \right) . \quad (\text{A29})$$

It can be seen that Equations A27 and A24 are the same, Equations A28 and A25 are the same, and Equations A29 and A26 are the same. By using Equations A4, A28, and A25,

$$x^2 + x^2 \left(\frac{-a^2 + b^2 + c^2}{a^2 - b^2 + c^2} \right) + x^2 \left(\frac{-a^2 + b^2 + c^2}{a^2 + b^2 - c^2} \right) = 1 ,$$

$$x = \frac{1}{\sqrt{1 + \left(\frac{-a^2 + b^2 + c^2}{a^2 - b^2 + c^2} \right) + \left(\frac{-a^2 + b^2 + c^2}{a^2 + b^2 - c^2} \right)}} . \quad (\text{A30})$$

Similarly

$$y = \frac{1}{\sqrt{1 + \left(\frac{a^2 - b^2 + c^2}{a^2 + b^2 - c^2} \right) + \left(\frac{a^2 - b^2 + c^2}{-a^2 + b^2 + c^2} \right)}} , \quad (\text{A31})$$

$$z = \frac{1}{\sqrt{1 + \left(\frac{a^2 + b^2 - c^2}{-a^2 + b^2 + c^2} \right) + \left(\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} \right)}} . \quad (\text{A32})$$

Inspection of Equations A30 , A31 , and A32 will show that their solution will result in real numbers only if the sum of any two of the quantities a^2 , b^2 , and c^2 exceeds the third quantity.

Appendix B

Triaxial Balancing of San Marco Flight Unit 2

The San Marco project is a joint program of Italy and the United States and the balancing of the Italian built spacecraft, performed by NASA at the Goddard Space Flight Center, was aided by the cooperation of personnel of the Italian Space Commission. Triaxial balancing was necessary for proper functioning of the drag measurement equipment on the spacecraft. San Marco flight unit 2 was balanced about its three reference axes by using stub shaft attachments to the spacecraft instead of the "wrap around" fixture used for earlier San Marco balance operations. The unit was balanced twice and both operations will be discussed here. The first operation reduced the dynamic unbalance to less than 25 oz.-in.² about all three axes; the second to less than 15 oz.-in.² Both operations reduced center of gravity displacement to less than 0.005 in.—the best value obtainable because of nonconcurrence of the reference axes.

First Balancing Operation

The stub shafts and spacecraft attachment adapters were individually statically balanced before assembly. Their static unbalance was minor and their design was assumed to preclude any appreciable dynamic unbalance. Alignment checks were made to ensure that the axes of rotation essentially coincided with the spacecraft reference axes. Deviations noted were considerably less than for the "wrap around" fixture used for flight unit 1 and did not exceed the practical limitations imposed by machining tolerances.

Measurements of initial static and dynamic unbalance about each reference axis were used to compute required corrections. Experience with unit 1 had shown that minor inconsistencies in measured data could affect the correction computation drastically, and even render the basic equations incapable of rational solution (as mentioned in the text). Therefore, a technique was developed to "normalize" (i.e., make consistent by averaging out the incompatibilities between redundant data) the measured data prior to calculation. The alternative solution of the basic set of dynamic balance equations was also derived and used and proved more amenable to arithmetic operations.

Corrections were applied to the spacecraft and residual unbalance was measured about all three axes. Results were: Spacecraft center of gravity displacement from reference axes intersection was reduced from approximately 0.2 in. to 0.003 in.; dynamic unbalance, axis XX, was reduced from 520 oz.-in.² to 25 oz.-in.²; dynamic unbalance, axis YY, was reduced from 1210 oz.-in.² to 22 oz.-in.²; dynamic unbalance, axis ZZ, was reduced from 1100 oz.-in.² to 20 oz.-in.².

The weight of the spacecraft (including shell) after balancing was 177.7 ± 0.1 lb. Approximately 8.5 lb (4.8 percent of the total) was added for balancing.

Second Balancing Operation

Between the first and second balancing operations the spacecraft had minor component modifications, but the initial unbalance was assumed to be small. Since the earlier dynamic unbalance correction was not optimized for minimum weight, it was decided to remove it, apply the equivalent optimized correction determined as described in the text, and then measure and correct the remaining unbalance. Substitution of the optimized for the original dynamic correction gave a net weight reduction of 0.95 lb, and unbalance was then measured as follows:

1. Static balance was 30 oz. in. about axis XX, 21 oz. in. about YY, and 27.5 oz. in. about ZZ.
2. Dynamic unbalance was 160 oz. in.² about axis XX, 124 oz. in.² about YY, and 139 oz. in.² about ZZ.

The apparent static unbalance was largely due to nonconcurrence of the actual axis of spin. For the spin axis it was noted that the spacecraft interface (at the separation plane) was a very loose fit on the adapter, having at least a 0.020 in. lateral slop, and was actually located (non-repeatably) by the external marmon clamp.

The dynamic unbalance was consistent with the expected results of modifications to the spacecraft; therefore the equivalence of the optimized to the original correction was essentially demonstrated.

The remaining dynamic unbalance did not warrant computing an optimized correction or consolidating the correction with existing weights; therefore a basic correction (no triaxially symmetric vector) was applied; this added 0.33 lb. Some final static balancing involved removal of 0.31 lb from existing balance weights, the overall operation reduced the total balancing weight on the spacecraft by 0.93 lb. Balancing required weight additions at 13 discrete locations, and because of spacecraft structural considerations 25 individual weights were actually attached. The final balance operation took 3 days and about 75 man-hours effort, much less than the earlier operation, because techniques had been improved and the spacecraft had less initial unbalance.

The residual unbalance obtained was as follows: Static unbalance was 8.6 oz. in. about spin axis XX, 9.7 oz. in. about YY, and 8.5 oz. in. about ZZ. (At least 70 percent of this static unbalance was the unavoidable consequence of nonconcurrence of the reference axes.) Dynamic unbalance was 10 oz. in.² about XX, 12 oz. in.² about YY, and 15 oz. in.² about ZZ. The final weight of the spacecraft, including shell and antennas, was 177.5 lb. About 4.25 percent of this was balance weight (considerably less than the 4.8 percent needed by the first balancing operation).

Conclusions

Entirely new techniques were developed and applied to achieve the desired balance of the San Marco. This was the most complex precision balancing operation yet performed. The minor

additional complications of concurrent control of inertias and total weight would have posed the ultimate complexity of complete control of all mass properties, so almost any balancing problem would be soluble by the general methods developed for San Marco.

The first balancing operation corrected unbalance with the minimum *number* of correction weights, and the second with a larger number of weights but with less total added weight.

The residual static unbalance was negligible because of the greater effect of nonconcurrence of reference axes. The residual dynamic unbalance, being equivalent to principal axis angular deviations between 0.15 and 0.02 degree, was low enough that non-orthogonality of reference axes may well have been significant.

Therefore, the San Marco flight unit 2 was balanced as well as the geometric errors of its own structure permitted. Effects due to unbalance could not have exceeded those due to geometric inaccuracies.

27/8/5